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A theory of the Casimir effect for compact regions

L.A. Manzoni[a], W.F. Wreszinski[b]

Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 - São Paulo, S.P., Brazil

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Abstract. We develop a mathematically precise framework for the Casimir effect. Our working hypothesis, verified in the case of parallel plates, is that only the regularization-independent Ramanujan sum of a given asymptotic series contributes to the Casimir pressure. As an illustration, we treat two cases: parallel plates, identifying a previous cutoff-free version (by Scharf and Wreszinski) as a special case, and the sphere. We finally discuss the open problem of the Casimir force for the cube. We propose an Ansatz for the exterior force and argue why it may provide the exact solution, as well as an explanation of the repulsive sign of the force.

1 A general framework for the Casimir effect

Significant progress on the Casimir effect from the experimental point of view occurred in recent times [1]. In spite of that, several theoretical problems remain, such as a real explanation of the sign of the force in the case of compact regions. The situation is worse with regard to a mathematically precise framework for the effect, due to the cutoff (or regularization) dependence of the energy, a fact emphasized by Hagen in [2] and somewhat less emphatically by Candelas in [3]. The physical reason why divergences occur is well understood [4] and is that the boundaries are treated by quantizing the radiation field with mode functions [5] which are adapted to the type of (classical) boundary conditions (b.c.), e.g., Dirichlet or Neumann. However, real boundaries consist of electrons and ions and such b.c. are not justified except if the particles act collectively in an essentially classical manner [5], which is a priori not the case [4], and our ignorance in dealing with this fact is signalled by divergences.

Divergences are, of course, well known in field theory, but they arise here in a different way, as explained above. Mathematical physicists, and several theoretical physicists, agree that a mathematically precise framework to cope with these divergences would be conceptually useful. Such frameworks exist in field theory (see [6,7] and references given there). A cutoff-free or "finite" theory of the Casimir effect (in the spirit of [7]) was attempted by Scharf and one of us (W.W.) in [8]. It requires, however, the use of periodic b.c., which are unphysical in the case of the electromagnetic field.

In this paper we present a thorough derivation of the results first announced in [9]. We reconsider the problem introducing ab initio an ultraviolet cutoff $(1/\Lambda)$. The Casimir energy (CE) $E_{\mathrm{vac}}(\Lambda)$ would diverge if the limit $\Lambda \to 0$ was taken, but we do not need to do so, because the Casimir *pressure* depends only on the Λ- independent term in the asymptotic expansion, which is the (RI) cutoff independent term of the Ramanujam sum of a divergent series. This idea is due to Dietz [10]. In Sect. 2 we show how the result of [8] for the parallel plates is recovered as a special case. Some of the ideas of [8] are also used and summarized below, for convenience.

Following [11], consider an electromagnetic field at $T = 0$ enclosed in cavities of identical shape, but made of different materials, the latter providing natural cutoffs for the high-frequency spectrum of zero point modes. The vacuum energy is thus given by

$$E_{\mathrm{vac}} = \frac{\hbar}{2} \sum_{\alpha} \omega_\alpha C_\alpha(\Lambda), \tag{1}$$

with $C_\alpha(\Lambda)$ material dependent cutoff functions dependent on a variable Λ with dimensions of length, which we normalize by

$$C_\alpha(\Lambda)|_{\Lambda=0} = 1. \tag{2}$$

Since E_{vac} has dimension (length)$^{-1}$ in natural units, it may be written as an (asymptotic) series

$$E_{\mathrm{vac}} = a_0 L^3 \Lambda^{-4} + a_1 L^2 \Lambda^{-3} + a_2 L \Lambda^{-2} + a_3 \Lambda^{-1} + a_4 L^{-1} + a_5 L^{-2} \Lambda + \dots, \tag{3}$$

where L is a length characterizing the spatial extension of the cavity. Dietz conjectured [10] that by a theorem of Ramanujan the Λ-independent term $a_4 L^{-1}$ in (3) is *independent of the regularization* (i.e., of the set $\{C_\alpha(\Lambda)\}$ in (1)) provided (2) holds. We shall return to this conjecture later.

In this paper we consider as in [8] the prototypical example of a massless scalar field confined in a compact re-

[a] e-mail: lmanzoni@fma.if.usp.br
[b] e-mail: wreszins@fma.if.usp.br

gion K (a compact manifold with boundary) – the modifications introduced by considering the full electromagnetic field will be mentioned later. We show that the ω_α in (1) should be identified with the eigenvalues of the *square root* of the Laplace–Beltrami operator. This is not unexpected, because the relativistic energy is $|\vec{k}| = (\vec{k}^2)^{1/2}$, but it has important consequences for the expansion (3). Consider [8] the field $A(x)$ quantized in infinite space

$$A(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 k}{\sqrt{2\omega}} \left[a(\vec{k}) \mathrm{e}^{-\mathrm{i}k\cdot x} + a^+(\vec{k}) \mathrm{e}^{\mathrm{i}k\cdot x} \right], \quad (4)$$

$$[A_-(x), A_+(y)] = \frac{1}{\mathrm{i}} D_0^{(+)}(x-y), \quad (5)$$

$$D_0^{(+)}(x) = \frac{\mathrm{i}}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2|\vec{k}|} \mathrm{e}^{-\mathrm{i}k\cdot x} = -\frac{\mathrm{i}}{4\pi^2} \frac{1}{(x_0 - \mathrm{i}0)^2 - \vec{x}^2}. \quad (6)$$

Time evolution is generated by the Hamiltonian $H = \int \mathrm{d}^3 x H(x)$, whose density can be written in the form

$$H(x) = \frac{1}{2} : \left[\left(\frac{\partial A}{\partial x_0} \right)^2 - A \frac{\partial^2 A}{\partial x_0^2} \right] : . \quad (7)$$

Normal ordering is defined in momentum space. In order to go over to a geometry with *boundaries*, we should formulate it in x-space by the point-splitting technique:

$$: \left(\frac{\partial A}{\partial x_0} \right)^2 : = \lim_{y\to x} : \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} : = \lim_{y\to x} \left\{ \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} + \frac{1}{\mathrm{i}} \frac{\partial^2}{\partial x_0^2} D_0^{(+)}(x-y) \right\}. \quad (8)$$

Finally,

$$H(x) = \lim_{y\to x} \left\{ \frac{1}{2} \frac{\partial A(x)}{\partial x_0} \frac{\partial A(y)}{\partial y_0} - \frac{1}{2} A(x) \frac{\partial^2 A(y)}{\partial y_0^2} + \frac{1}{\mathrm{i}} \frac{\partial^2}{\partial x_0^2} D_0^{(+)}(x-y) \right\}. \quad (9)$$

Taking into account that real boundaries consist of electrons and ions and the field which interacts with them is quantized in *infinite space*, we consider (9) to be the Hamiltonian density describing the field both free and with boundaries. In the latter case, however, the first two terms in (9) must be defined in the adequate Fock space, i.e., the concrete representation of the field operator is dictated by the geometry. Consider a compact region K and Dirichlet b.c. $A(x) = 0$ for $\vec{x} \in \partial K$. Then $A(x)$ may be expanded as follows:

$$A(x) = \sum_n \frac{1}{\sqrt{2\omega_n}} \left[a_n u_n(\vec{x}) \mathrm{e}^{-\mathrm{i}\omega_n x_0} + a_n^+ u_n(\vec{x}) \mathrm{e}^{\mathrm{i}\omega_n x_0} \right], \quad (10)$$

where u_n are normalized real eigenfunctions of the Laplacian in K, satisfying Dirichlet or Neumann b.c. (discrete spectrum):

$$-\Delta u_n(\vec{x}) = \omega_n^2 u_n(\vec{x}). \quad (11)$$

The concrete Fock representation is now specified by considering a_n^+, a_n as emission and absorption operators ($[a_n, a_m^+] = \delta_{nm}$) and defining the vacuum by

$$a_n \Omega = 0, \quad \forall n. \quad (12)$$

We thus find in this Fock representation

$$H(x) = \frac{1}{2} ; \left(\frac{\partial}{\partial x_0} A(x) \right)^2 ; - \frac{1}{2} ; A(x) \frac{\partial^2}{\partial x_0^2} A(x); + \frac{1}{\mathrm{i}} \lim_{y\to x} \frac{\partial^2}{\partial x_0^2} \left\{ D_0^{(+)}(x-y) - D_K^{(+)}(x,y) \right\}; \quad (13a)$$

where

$$D_K^{(+)}(x_0 - y_0, \vec{x}, \vec{y}) = \mathrm{i} \sum_n \frac{1}{2\omega_n} u_n(\vec{x}) u_n(\vec{y}) \mathrm{e}^{-\mathrm{i}\omega_n (x_0 - y_0)}, \quad (13b)$$

and the semicolons in (13a) denote normal ordering with respect to the new emission and absorption operators a_n^+ and a_n. Notice that $D_0^{(+)}$ is the solution of the wave equation $\Box D_0^{(+)} = 0$ with initial conditions

$$D_0^{(+)}(+0, \vec{x}) = \frac{\mathrm{i}}{4\pi^2} \frac{1}{\vec{x}^2 + \mathrm{i}0}, \quad (14a)$$

$$\left(\partial_0 D_0^{(+)} \right)(0, \vec{x}) = \frac{1}{2} \delta(\vec{x}). \quad (14b)$$

$D_0^{(+)}(+0, \vec{x})$ is the Green's function of the square root of $-\Delta$ in infinite space [8]. Similarly

$$\left(\partial_0 D_K^{(+)} \right)(x, y) \Big|_{y_0 = x_0} = \frac{1}{2} \delta(\vec{x} - \vec{y}), \quad (15a)$$

$$D_K^{(+)}(+0, \vec{x}, \vec{y}) = \frac{\mathrm{i}}{2} (-\Delta_K)^{-1/2} (\vec{x} - \vec{y}), \quad (15b)$$

where Δ_K denotes the Laplacian on K, with Dirichlet or Neumann b.c.

We now consider two types of cutoff functions, one of them general, satisfying (2), the other special, of type

$$C_\alpha(\Lambda) = C(\Lambda \omega_\alpha), \quad (16a)$$

satisfying

$$C(0) = 1. \quad (16b)$$

We shall also be interested in a particular case of (16a), namely

$$C(k) = \mathrm{e}^{-\Lambda k}, \quad k \geq 0. \quad (17)$$

In terms of the special choice (17), we may, by (9) and (12)–(13b), compute a regularized vacuum energy density $H_{\mathrm{vac}}(x, \Lambda)$ in the following way:

$$H_{\mathrm{vac}}(x, \Lambda) = \frac{1}{2} \frac{\partial}{\partial \Lambda} \left\{ \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k \ \mathrm{e}^{-\mathrm{i}[k_0 \tau - \vec{k}\cdot(\vec{x} - \vec{y})]} \Big|_{\substack{\vec{y}=\vec{x} \\ \tau=0}} \right.$$

$$\times\, C(k_0) - \sum_n \left[u_n(\vec{x})\right]^2 C(\omega_n) \Bigg\}. \tag{18}$$

As an aside, notice that (17) corresponds to the ascription of a small imaginary part $-\mathrm{i}\Lambda$ to $x_0 - y_0 = \tau$, and thus represents a "natural" choice, akin to the principal value in distribution theory [12]. For this special case (17) becomes

$$H_{\text{vac}}(\vec{x},\Lambda) = \frac{1}{2}\frac{\partial}{\partial \Lambda}\left[P(\vec{x},\vec{x};\Lambda) - P_0(\vec{x},\vec{x};\Lambda)\right], \tag{19a}$$

where P, P_0 satisfy the "heat equation"

$$\left(\frac{\partial}{\partial \Lambda} - (-\Delta_{\vec{x}})^{1/2}\right) P(\vec{x},\vec{y};\Lambda) = 0, \tag{19b}$$

with the b.c.

$$P(\vec{x},\vec{y};\Lambda) = 0 \quad \text{if } \vec{x} \quad \text{or } \vec{y} \in \partial K, \tag{19c}$$

in the case of Dirichlet b.c.

There exist methods to compute the asymptotic expansion (in Λ) of the quantity in brackets in (19a) [13], which solve the problem in principle, but the actual form (3), with the given coefficients, depends on the details of the discrete (eigenvalue) spectrum of $(-\Delta)^{1/2}$.

Let now L be a linear dimension of the compact region $K \equiv K_L$ and M a linear dimension of a region K_M of which K_L is a subset. Typically, if K_L is a cube of side L, K_M is a cube of side $M > L$ concentric with K_L, and similarly for a sphere or other manifolds. It is correct to impose the same b.c. (e.g. Dirichlet or Neumann) on K_M in order to define the outer Casimir problem [14,11]. In fact, previous work on the sphere using the Sommerfeld radiation condition was not correct, although the results were right, because it did not lead to real eigenvalues [15]. Define

$$E_{\text{vac}}(L,\Lambda,M) = E_{\text{vac}}^{\text{inner}}(L,\Lambda) + E_{\text{vac}}^{\text{outer}}(L,\Lambda,M), \tag{20}$$

where

$$E_{\text{vac}}^{\text{inner}}(L,\Lambda) \equiv \int_{K_L} \mathrm{d}^3x H(\vec{x},\Lambda), \tag{21}$$

and

$$E_{\text{vac}}^{\text{outer}}(L,\Lambda,M) = \int_{K_M\setminus K_L} \mathrm{d}^3x \tilde{H}(\vec{x},\Lambda). \tag{22}$$

As previously remarked, if Dirichlet b.c. are imposed on K_L, H (respectively $\tilde{H}$) is the density (18) with the $\{u_n\}$ defined by Dirichlet b.c. imposed on K_L (respecively K_L and K_M). Analogous definitions hold for other b.c. (e.g. Neumann or mixed). If (1) and (2) are adopted, the second sum in (20) refers, then, to the modes ω_n corresponding to the solution of (11) in $K_M\setminus K_L$, with the above-mentioned b.c. Suppose that both $E_{\text{vac}}^{\text{inner}}(L,\Lambda)$ and $E_{\text{vac}}^{\text{outer}}(L,\Lambda,M)$ have an asymptotic series (3), and let $E_{\text{vac}}^{\text{inner}}(L)(\equiv a_4^{\text{inner}}/L)$ and $E_{\text{vac}}^{\text{outer}}(L,M)$ be the corresponding Λ-independent terms. Then the Casimir pressure on the boundary surface $p_{\mathrm{C}}(L)$ (a measurable quantity) is defined by the thermodynamic formulae (zero absolute temperature):

$$p_{\mathrm{C}}(L) = p_{\mathrm{C}}^{\text{inner}}(L) - p_{\mathrm{C}}^{\text{outer}}(L), \tag{23a}$$

where the relative minus sign takes into account that $p_{\mathrm{C}}^{\text{outer}}$ refers to a normal vector pointing inwards towards K_L, while $p_{\mathrm{C}}^{\text{inner}}$ refers to a normal vector pointing outward, and

$$p_{\mathrm{C}}^{\text{inner}}(L) = -\frac{\partial E_{\text{vac}}^{\text{inner}}(L)}{\partial V_{\text{inner}}(L)}, \tag{23b}$$

$$p_{\mathrm{C}}^{\text{outer}}(L) = -\lim_{M\to\infty}\frac{\partial E_{\text{vac}}^{\text{outer}}(L,M)}{\partial V_{\text{outer}}(L,M)}, \tag{23c}$$

and an important feature of the *thermodynamic limit* [16] is that the derivative in (23c) is taken with M *fixed*; only L varies.

It is essential that the CE be independent of the cutoff function C in (1) or (16a) provided it satisfies (2) or (16b). As remarked in [10], a necessary condition for this regularization independence (RI) to hold is that (3) contain no logarithmic terms, because, otherwise, the "Λ-independent term" is obviously ill defined. For the cube there are no such terms in (3), but such is not the case for the sphere; however such terms may be omitted in the case of the sphere because they cancel in the expression $E_{\text{vac}}^{\text{inner}}(L,\Lambda) + E_{\text{vac}}^{\text{outer}}(L,\Lambda,M)$ which have an asymptotic series (3) as $M \to \infty$, so that for the sphere of radius a we have

$$p_{\mathrm{C}}(a) = -\frac{1}{4\pi a^2}\frac{\partial}{\partial a}\frac{a_4^{\text{sphere}}}{a} = \frac{a_4^{\text{sphere}}}{4\pi a^4}. \tag{24}$$

A full proof of RI is given in Sect. 2 for parallel plates. We shall leave a more detailed discussion of higher dimensional cases [17] to a further publication, but we wish to make a few important remarks.

(a) the Λ-independent term in (3) should coincide with the Λ-independent term of the Ramanujan sum of a divergent series of positive terms, such as (1), with $C_\alpha(\Lambda) \equiv 1$ see [18] (p. 318 ff.) and Sect. 1. According to this concept, for instance

$$1 + 1 + \cdots + 1 + \cdots = -\frac{1}{2}(\Re, 0),$$
$$1 + 2 + 3 + \cdots = -\frac{1}{12}(\Re, 0),$$

taking the origin as reference point (see [18], 13.10.11). This is proved in Sect. 2 for parallel plates, and is the basis of RI;

(b) the present definition of the CE is mathematically rigorous. In particular, the limit $\Lambda \to 0$ is never taken. In fact, (3) shows that, in general, it does not exist (an exception is the Casimir effect for parallel plates with periodic b.c., see [8] and Sect. 2). The reason for this is that we do not know how to treat the surface properly in microscopic terms, a formidable problem (see the conclusion);

(c) RI justifies the definition of the Λ-independent term in (3) as the CE physically: it reflects the field theoretic structure of the vacuum state which is independent of the cavity materials [10]. It is also expected to be the only term in (3) which contributes to the *pressure*: this was proved in [10] for parallel plates.

Section 2 entails a complete proof of RI for parallel plates (a "limit" of a compact region), as well as the explanation of the "theory without cutoffs" for the case of periodic boundary conditions in [8]. The sphere is also treated as an illustration, in Sect. 3. In contrast to the parallel plates, the force for the sphere is repulsive, as known since the pioneering work of Boyer [11]. The nature of our derivation, which is an adaptation of the ideas and results of [15,14] to our framework, does not convey an intuitive "explanation" of the sign of the force. This is a difficult problem because the CE is a sum of *fluctuations* of the electric and magnetic fields in the vacuum state. A basic issue is: if the flat parallel plate geometry is changed to a compact manifold with boundary, how does the sign of the force change and why? This question is most clearly analyzed in the case of the cube, which is the simplest deformation of the parallel plates geometry. In Sect. 4 we consider the *interior* problem for the cube, using the method of the Poisson summation formula used in [8]. This method had already been used for the same purpose in [19]. Since this reference is not readily available, we include our (independent) derivation which generalizes [19] in the sense that we obtain the full asymptotic formula, and we see that it fits nicely into the present framework. It should also be remarked that it coincides with the numerical result of [20]. The inner problem leads, however, to an *attractive* force, while the result for the sphere leads us to expect a repulsive force. Therefore, the repulsive nature must be due entirely to the *exterior* pressure.

In Sect. 5 we introduce an Ansatz to solve the exterior problem for the cube, which leads to a repulsive force. We also have applied the Ansatz to the only known soluble case with flat geometry, i.e., the case of parallel plates (Appendix A). From the analysis of this soluble case we identify the physical reason why our Ansatz does not modify the pressure: the Ansatz introduces some extra stresses, but these are *parallel* to the plates (faces of the cube) so that the pressure *on* the plates (faces) is insensitive to these additional stresses, then providing the correct result for the plates and (we believe) for the cube. If the latter conjecture is true, an "explanation" of the sign of the force also follows. This is left to the conclusion and open problems in Sect. 6.

We have used a very general class of cutoffs in *momentum* space for parallel plates. For the cube the proof is essentially the same as for parallel plates, but there are some subtleties in the case of the sphere which have not yet been fully worked out. Nevertheless, it is an *open problem* whether only the cutoff-independent part of the CE is relevant to the *pressure*, except in the explicit case of parallel plates [10]. We shall admit this as a working hypothesis throughout.

2 Parallel plates

We consider the problem of parallel plates, with distance d along the z-axis; take the positions of the plates at $z = 0$ and $z = d$, and adopt the form (16a) in (18), (20) with Dirichlet b.c. (Neumann b.c. yield the same results). The *inner* Casimir problem corresponds to the region $K_L = K_d = \{\vec{x} \in \mathbb{R}^2 \times [0, d]\}$, and the *outer* one to the region $K_R \backslash K_L = \{\vec{x} \in \mathbb{R}^2 \times [d, d+R]\} \cup \{\vec{x} \in \mathbb{R}^2 \times [-R, 0]\}$. The eigenfunctions associated to the inner problem are

$$u_n^{\text{inner}}(k_x, k_y) = \frac{1}{2\pi}\sqrt{\frac{2}{d}} \sin\left(\frac{n\pi}{d} z\right) \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}, \quad n = 1, 2, 3, \cdots, \tag{25}$$

corresponding to the eigenvalues of $(-\Delta)^{1/2}$ given by

$$\omega_{n,k_x,k_y}^{\text{inner}} = \sqrt{\left(\frac{n\pi}{d}\right)^2 + k_x^2 + k_y^2}, \tag{26}$$

in (11). The outer eigenfunctions are

$$\begin{aligned} u_n^{\text{outer},1}(k_x, k_y) &= \frac{1}{2\pi}\sqrt{\frac{2}{R}} \sin\left(\frac{n\pi}{R}(z-d)\right) \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}, \\ u_n^{\text{outer},2}(k_x, k_y) &= \frac{1}{2\pi}\sqrt{\frac{2}{R}} \sin\left(\frac{n\pi}{R} z\right) \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}, \end{aligned} \tag{27}$$

with eigenvalues

$$\omega_{n,k_x,k_y}^{\text{outer}} = \sqrt{\left(\frac{n\pi}{R}\right)^2 + k_x^2 + k_y^2}. \tag{28}$$

We first adopt the choice (17). Introducing polar coordinates in the x–y plane, we calculate the first (inner) sum in (20) (we do *not* integrate along $(x, y) \in \mathbb{R}^2$, which would yield $+\infty$). The proper way to do this is to limit the x–y-plane integration to a finite region with area A, and then take the limit for $\mathcal{E} = E/A$ (this procedure yields the same results as presented here and we omit it for brevity):

$$\mathcal{E}_{\text{vac}}^{\text{inner}}(\Lambda, d) = \frac{1}{2(2\pi)^2}\left\{ -2d \int_0^\infty \mathrm{d}k k^3 \mathrm{e}^{-\Lambda k} + 2\pi \sum_{n=1}^\infty \int_0^\infty \mathrm{d}k k \mathrm{e}^{-\Lambda\sqrt{(n\pi/d)^2+k^2}} \sqrt{\left(\frac{n\pi}{d}\right)^2 + k^2}\right\}. \tag{29}$$

Performing the change of variable $k_n' = ((n\pi/d)^2 + k^2)^{1/2}$ in the second integral in the r.h.s. of (29) we obtain

$$\begin{aligned} \mathcal{E}_{\text{vac}}^{\text{inner}}(\Lambda, d) &= \frac{1}{2(2\pi)^2}\left\{ -2d \int_0^\infty \mathrm{d}k k^3 \mathrm{e}^{-\Lambda k} + 2\pi \sum_{n=1}^\infty \int_{n\pi/d}^\infty \mathrm{d}k_n' k_n'^2 \mathrm{e}^{-\Lambda k_n'} \right\} \\ &= \frac{d}{(2\pi)^2}\left\{ -6\Lambda^{-4} + \frac{\partial^2}{\partial \Lambda^2}\left[\frac{1}{\Lambda^2} \frac{\frac{\Lambda\pi}{d}}{\mathrm{e}^{\Lambda\pi/d} - 1}\right]\right\}, \end{aligned} \tag{30}$$

we now use the expansion ([18], p. 320) in (30)

$$\frac{t}{\mathrm{e}^t-1}=1-\frac{1}{2}t+\sum_{k=1}^{\infty}(-1)^{k-1}B_k\frac{t^{2k}}{(2k)!}, \tag{31}$$

obtaining ($B_2=1/30$):

$$\mathcal{E}_{\mathrm{vac}}^{\mathrm{inner}}(\Lambda,d)=-\frac{1}{4\pi\Lambda^3}-\frac{1}{2}\frac{\pi^2}{720d^3}+\mathcal{O}(\Lambda), \tag{32}$$

and thus

$$\mathcal{E}_{\mathrm{Casimir}}^{\mathrm{inner}}=-\frac{1}{2}\frac{\pi^2}{720d^3}. \tag{33}$$

Two remarks are in order. The surface term $-(1)/(4\pi\Lambda^3)$ in (32) is absent for periodic b.c., because the latter allow the term $n=0$ in (29) which exactly cancels it. This explains the result of [8]. The external CE is *zero* due to (27) and (28) because, for the outer problem, d in (32) is replaced by R, and thus in the limit $R\to\infty$

$$\mathcal{E}_{\mathrm{Casimir}}^{\mathrm{outer}}=0. \tag{34}$$

Finally,

$$\mathcal{E}_{\mathrm{Casimir}}=-\frac{1}{2}\frac{\pi^2}{720d^3}. \tag{35}$$

The above energy is one half of the result for the electromagnetic field, due to the summation over the two polarization states in the latter. Notice also that, in natural units, $\mathcal{E}$ is of order $(\text{length})^{-3}$.

An amusing aspect of the present derivation is that it seems to depend on the choice (17), i.e., of an exponential cutoff in (29) and (30), which, due to (31), leads to (32). Consider now a general cutoff function (16a). Omitting the volume term in (29), we may write

$$\mathcal{E}_{\mathrm{vac}}^{\mathrm{inner}}(\Lambda,d)=\lim_{n\to\infty}\frac{1}{8\pi}\sum_{m=1}^{n}g(m), \tag{36}$$

where

$$\begin{aligned}g(m)&=\int_0^\infty \mathrm{d}u\sqrt{u+\left(\frac{m\pi}{d}\right)^2}C\left(\Lambda\sqrt{u+\left(\frac{m\pi}{d}\right)^2}\right)\\&=\int_{(m\pi/d)^2}^\infty \mathrm{d}u\sqrt{u}C(\Lambda\sqrt{u}).\end{aligned} \tag{37}$$

It is of interest to compute

$$\frac{d}{\pi}g^{(1)}(m)=-2\left(\frac{m\pi}{d}\right)^2C\left(\Lambda\frac{m\pi}{d}\right), \tag{38a}$$

$$\frac{d}{\pi}g^{(2)}(m)=-4\frac{\pi}{d}\left(\frac{m\pi}{d}\right)C\left(\Lambda\frac{m\pi}{d}\right)-2\left(\frac{m\pi}{d}\right)^2\left(\frac{\Lambda\pi}{d}\right)C^{(1)}\left(\Lambda\frac{m\pi}{d}\right), \tag{38b}$$

$$\begin{aligned}\frac{d}{\pi}g^{(3)}(m)=&-4\left(\frac{\pi}{d}\right)^2C\left(\Lambda\frac{m\pi}{d}\right)\\&-8\frac{\pi}{d}\left(\frac{m\pi}{d}\right)\left(\frac{\Lambda\pi}{d}\right)C^{(1)}\left(\Lambda\frac{m\pi}{d}\right)\\&-2\left(\frac{m\pi}{d}\right)^2\left(\frac{\Lambda\pi}{d}\right)^2C^{(2)}\left(\Lambda\frac{m\pi}{d}\right).\end{aligned} \tag{38c}$$

By [18] (p. 326), under the following conditions (44) and (45) on C:

$$\sum_{m=1}^{n}g(m)-\frac{2d}{\pi}\int_0^\infty \mathrm{d}qq^3C(\Lambda q)+\frac{1}{2}g(0)\underset{n\to\infty}{\longrightarrow}\Sigma_k, \tag{39}$$

where

$$\Sigma_k=-S_k(0)-\frac{1}{(2k+2)!}\int_0^\infty\psi_{2k+2}(t)g^{(2k+2)}(t)\mathrm{d}t, \tag{40}$$

and

$$\psi_k(x)=\phi_k(x)\bmod 1\quad(\text{i.e., equal to }\phi_k(x)\text{ for }0\le x<1\text{ with period }1), \tag{41}$$

and ϕ_k are defined by

$$t\frac{\mathrm{e}^{xt}-1}{\mathrm{e}^t-1}=\sum_{n=1}^{\infty}\phi_k(x)\frac{t^n}{n!}, \tag{42}$$

and

$$S_k(0)=\sum_{r=1}^{k}(-1)^{r-1}\frac{B_r}{(2r)!}g^{(2r-1)}(0). \tag{43}$$

We changed the notation of [18]: the C_k on p. 326 corresponds to our Σ_k. Notice that the second term in (39) corresponds to the subtraction of the vacuum term, which appears in a natural way as a necessary requirement in a purely mathematical context! The term $(1/2)g(0)$ contributes only to the Λ-dependent terms in the asymptotic series. Let us now write down the following theorem.

Theorem. Let the special cutoff function of type (16a) satisfy, besides (16b), the conditions: C is infinitely differentiable and its derivatives $C^{(k)}$ ($C^{(0)}\equiv C$) satisfy

$$\int^\infty C^{(k)}(x)\mathrm{d}x<\infty, \tag{44}$$

$$C^{(k)}(x)\underset{x\to\infty}{\longrightarrow}0. \tag{45}$$

Then, for Dirichlet (or Neumann) b.c. the Λ-independent term in (3) is the cutoff-independent part of the Ramanujan sum of the divergent series (1) with $C_\alpha(\Lambda)\equiv 1$, and is therefore RI, i.e., independent of C.

Remark. Σ_k ($k\ge 1$) is referred to as the $(\Re,0)$ sum of the (divergent) series $\sum_{m=1}^\infty g(m)$, where $\Re$ refers to Ramanujan and 0 to the reference point (the origin in our case). Usually (see, e.g., [21], p. 138), the result is presented informally without the important last term in (40), and assuming that C satisfies $C^{(k)}(0)=0$ for all $k\ge 1$, besides (16b), which is not satisfied by the special choice (17) (see, however, [22] for a nice approach to the subject).

Proof. The fact that Σ_k is independent of k for $k\ge 1$ follows from [18] (pp. 326 ff). Choose $k=2$. By (40)–(43),

$$\Sigma_2=-\frac{B_1}{2}g^{(1)}(0)+\frac{B_2}{24}g^{(3)}(0)-\frac{1}{6!}\int_0^\infty\psi_6(t)g^{(6)}(t)\mathrm{d}t. \tag{46}$$

Putting (38a), (38c) and (16b) into (46), we find

$$\Sigma_2 = -\frac{B_2}{6}\left(\frac{\pi}{d}\right)^3 + \mathcal{O}(\Lambda^2), \tag{47}$$

which leads to (35) by (36). The term $\mathcal{O}(\Lambda^2)$ in (47) comes from $g^{(6)}$, making the change of variable $t' = (\Lambda\pi/d)t$ in the integral in (46) and taking into account that ψ_k is $\mathcal{O}(1)$.

What if we choose $k = 1$? By (43) and (38a), $S_1(0) = 0$, but, in (40), we still have the second term

$$\Sigma_1 = -\frac{1}{24}\int_0^\infty \psi_4(t) g^{(4)}(t)\mathrm{d}t. \tag{48}$$

We use the recurrence ([18], 13.2.13)

$$\psi^{(1)}_{2m+1} = (2m+1)\left\{\psi_{2m} + (-1)^{m-1}B_m\right\}, \tag{49}$$

with $m = 2$, obtaining

$$\psi_4 - B_2 = \frac{1}{5}\psi^{(1)}_5, \tag{50}$$

which we insert in (48), getting

$$\Sigma_1 = -\frac{1}{24}\int_0^\infty \frac{\psi^{(1)}_5(t)}{5} g^{(4)}(t)\mathrm{d}t - \frac{1}{24}B_2\int_0^\infty g^{(4)}(t)\mathrm{d}t. \tag{51}$$

Integration by parts in the first term on the r.h.s. of (51) and use of (38c) in the second term yield (using $\psi_n(0) = 0$)

$$\Sigma_1 = \frac{1}{120}\int_0^\infty \psi_5(t) g^{(5)}(t)\mathrm{d}t + \frac{B_2}{24}g^{(3)}(0). \tag{52}$$

A further integration by parts using the recurrence ([18], 13.2.13)

$$\psi^{(1)}_{2m} = 2m\psi_{2m-1}, \tag{53}$$

brings (52) to the form (46). We have thus proved

$$\Sigma_k = -\frac{B_2}{6}\left(\frac{\pi}{d}\right)^3 + \mathcal{O}(\Lambda^2), \tag{54}$$

for all $k \geq 1$ (the present argument is easily generalized). Thus, for parallel plates and Dirichlet b.c. the Λ-independent term in the asymptotic series (3) is regularization independent and is the $(\Re, 0)$ sum of the divergent series (36). Neumann b.c. yield the same result.

3 The sphere

The Casimir effect for b.c. on a sphere was first considered in the classic paper by Boyer [11] and since then it has been considered from various viewpoints: source theory [25], multiple scattering [26], dimensional dependence of the effect [27] as well as an improved mode summation method [15,14] (see also [28]). In [31] it is shown how a natural subtraction method ensures convergence of the mode sum, and in [32] RI for the ball has been proved (for a more detailed reference list, see [29,22]).

Here we will to reconsider the CE for a massless scalar field subjected to Dirichlet b.c. on a sphere in the light of the above developed theory. We will consider the original sphere, of radius a, embedded in a concentric greater sphere of radius $R > a$. As is well known, for the sphere it is convenient to consider the inner and outer regions together in order to avoid logarithmic contributions for the CE. So, taking into account the $(2l+1)$-fold degeneracy of each eigenvalue, we have

$$E_{\mathrm{vac}} = \sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\sum_{n=1}^{\infty}\omega_{nl}C_{nl}(\Lambda), \tag{55}$$

where ω_{nl} are the eigenfrequencies. The sum over n in (55) can be changed into an integral by using the Cauchy theorem [15,14,28]. Here we will follow [14] with an crucial difference: the cutoff functions used in [14], while appropriate to treat the electromagnetic field, do not render the integrals well defined in the massless scalar field case, so that we will adopt[1]

$$C_{nl}(\Lambda) = \mathrm{e}^{-\Lambda(\nu/a + \omega_{nl})}, \tag{56}$$

for the cutoff functions, which satisfies the normalization condition (2). Besides this, it is important to analyze the asymptotic behavior of E_{vac} based on more general cutoff functions.

Then, we can rewrite (55) as [15,14]

$$E_{\mathrm{vac}} = -\frac{1}{a}\sum_{l=0}^{\infty} Q_l, \tag{57}$$

with $(\nu = l + (1/2))$

$$Q_l = \frac{\nu^2}{\pi}\mathrm{e}^{-\Lambda\nu/a}\mathrm{Re}\,\mathrm{e}^{-\mathrm{i}\varphi} \times \int_0^\infty y \exp\left\{-\mathrm{i}\nu\frac{\Lambda}{a}y\mathrm{e}^{-\mathrm{i}\varphi}\right\}\frac{\mathrm{d}}{\mathrm{d}y}\ln f_l(\mathrm{i}\nu y\mathrm{e}^{-\mathrm{i}\varphi})\mathrm{d}y, \tag{58}$$

where φ is an (small) angle which gives a sense of orientation to the contour of integration with respect to the imaginary axis of z (see [14]), and

$$f_l(\mathrm{i}z) = -\frac{1}{z}I_\nu(z)K_\nu(z). \tag{59}$$

Now using the uniform asymptotic expansions for the Bessel functions I_ν and K_ν [30] we can obtain an asymptotic expansion for Q_l which is valid for large orders. Then, in general, we can rewritte E_{vac} as [15]

$$E_{\mathrm{vac}} = E_{\mathrm{asym}} - \frac{1}{a}\sum_{l=0}^{n}\Delta Q_l, \tag{60}$$

where E_{asym} stands for the expression obtained from (57) and (58) by using the asymptotic expansions for the Bessel

[1] More general cutoffs of the above type have been used by Hagen in a different context [2]

functions [30], $\Delta Q_l = Q_l - Q_l^{\rm asym}$, and n is such that for $l > n$ the asymptotic expansion $Q_l^{\rm asym}$ affords a good approximation for Q_l (i.e., $\Delta Q_l \simeq 0$ for $l > n$).

Then, we obtain (after performing a rotation of the integration contour $y\mathrm{e}^{-\mathrm{i}\varphi} \to y$)

$$\begin{aligned} E_{\rm asym} = &-\frac{2}{\pi}\frac{a^2}{\Lambda^3}\mathrm{Re}\int_0^\infty \frac{y}{(1+iy)^3}\frac{\mathrm{d}}{\mathrm{d}y}\ln t\mathrm{d}y \\ &- \frac{1}{\pi}\frac{1}{\Lambda}\mathrm{Re}\int_0^\infty \frac{y}{(1+iy)}\frac{\mathrm{d}}{\mathrm{d}y}\alpha(t)\mathrm{d}y \\ &- \frac{1}{\pi a}\zeta\left(2,\frac{1}{2}\right)\mathrm{Re}\int_0^\infty y\frac{\mathrm{d}}{\mathrm{d}y}\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right]\mathrm{d}y \\ &+ \mathcal{O}(\Lambda), \end{aligned} \tag{61}$$

where $\zeta(s,a) = \sum_{l=0}^{\infty}(l+a)^{-s}$ is the Hurwitz zeta function. From this expression must be clear why we have introduced the cutoff functions (56) rather than $\mathrm{e}^{-\Lambda\omega_{nl}}$ used in [14]. Namely, in the absence of the term $\mathrm{e}^{-\Lambda\nu/a}$ in (56) the first integral in (61) would have a non-integrable singularity in the origin, but all integrals are well defined if we adopt (56).

From (61) we have $\left(\zeta\left(2,1/2\right) = \pi^2/2\right)$

$$E_{\rm asym} = -\frac{a^2}{8\Lambda^3} - \frac{5}{1024\Lambda} + \frac{35\pi^2}{65536a} + \mathcal{O}(\Lambda). \tag{62}$$

Now, it remains to calculate $\sum_{l=0}^{n}\Delta Q_l$ in (60). Notice that in this term the sum is finite and we do not have any divergence. Then, since $\varphi > 0$ may be considered a small angle ($\sin\varphi > 0$ and $\cos\varphi > 0$) we may integrate (58) by parts and perform a rotation of the integration contour ($y\mathrm{e}^{-\mathrm{i}\varphi} \to y$) to obtain

$$Q_l = -\frac{\nu}{\pi}\int_0^\infty \mathrm{d}y \ln\left[2yI_\nu(y)K_\nu(y)\right] + \mathcal{O}(\Lambda), \tag{63}$$

which is nothing but the Q_l in [15] (except for a sign). So we may take advantage of the numerical results in [15] for this expression.

Analogously, we may obtain a expression for $Q_l^{\rm asym}$, appropriate for when there is no infinite summation, given by

$$\begin{aligned} Q_l^{\rm asym} = &-\frac{\nu^2}{\pi}\int_0^\infty \mathrm{d}y\frac{\mathrm{d}}{\mathrm{d}y}\ln\left[\frac{y}{\sqrt{1+y^2}}\right] - \frac{1}{\pi}\int_0^\infty \mathrm{d}y\alpha(t) \\ &- \frac{1}{\pi\nu^2}\int_0^\infty \mathrm{d}y\left[\beta(t)-\frac{1}{2}\alpha^2(t)\right] + \mathcal{O}(\Lambda), \end{aligned} \tag{64}$$

which after integration yields

$$Q_l^{\rm asym} = \frac{\nu^2}{2} + \frac{1}{128} - \frac{35}{32768\nu^2} + \cdots. \tag{65}$$

Then, we can take $n = 4$ in (60) as a good approximation (see [15]) obtaining

$$E_{\rm vac} = -\frac{a^2}{8\Lambda^3} - \frac{5}{1024\Lambda} + \frac{0.002819}{a} + \cdots, \tag{66}$$

which yields $a_4^{\rm sphere} \simeq 0.002819$ for the coefficient of the Λ-independent term in the asymptotic series (3) for E. Therefore the CE is

$$E_{\rm Casimir} = \frac{a_4^{\rm sphere}}{a} \simeq \frac{0.002819}{a}, \tag{67}$$

and by (24) we see that the Casimir force for massless scalar field with Dirichlet b.c. on a sphere is repulsive. This result was obtained with greater precision in [27] (also see [14,15,33]).

While the numerical result provided by (67) is not new, the above calculation illustrates the fact that when we use more general cutoffs like (56) (which in the present case is mandatory) we are faced with an asymptotic series in Λ for $E_{\rm vac}$; see (3) and (66). Then, the method discussed above provides an unambiguous way to identify the CE.

4 The interior problem for the cube

Consider now a cube K of side L, with Dirichlet b.c. (Neumann b.c. may be handled analogously). The normalized eigenfunctions and eigenvalues of $(-\Delta)^{1/2}$ are

$$\begin{aligned} &u_{n_1n_2n_3}(\vec{x}) \\ &= \left(\frac{2}{L}\right)^{3/2}\sin\left(\frac{n_1\pi x_1}{L}\right)\sin\left(\frac{n_2\pi x_2}{L}\right)\sin\left(\frac{n_3\pi x_3}{L}\right), \\ &(-\Delta)^{1/2}u_{n_1n_2n_3}(\vec{x}) \\ &= \frac{\pi}{L}|\vec{n}|u_{n_1n_2n_3}(\vec{x}), \quad |\vec{n}| = (n_1^2+n_2^2+n_3^2)^{1/2}, \end{aligned} \tag{68}$$

where $n_i = 1, 2, \cdots$ $(i = 1, 2, 3)$.

We consider

$$E_{\rm vac}(\Lambda) = \int_K \mathrm{d}^3x H(\vec{x},\Lambda). \tag{69}$$

By (13a),

$$E_{\rm vac}(\Lambda) = \frac{1}{2}\frac{\partial}{\partial\Lambda}\left\{\frac{L^3}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{e}^{-\Lambda|\vec{k}|} - \sum_{\vec{n}}\mathrm{e}^{-\Lambda\omega_{\vec{n}}}\right\}. \tag{70}$$

By (68), $\omega_{\vec{n}} = (\pi/L)|\vec{n}|$ and hence

$$\begin{aligned} E_{\rm vac}(\Lambda) = &-\frac{3}{2\pi^2}L^3\Lambda^{-4} \\ &- \frac{1}{16}\frac{\partial}{\partial\Lambda}\left[\sum_{\vec{n}\in\mathbb{Z}^3}\mathrm{e}^{-a|\vec{n}|} - 3\sum_{\vec{n}\in\mathbb{Z}^2}\mathrm{e}^{-a|\vec{n}|} + 3\sum_{n\in\mathbb{Z}}\mathrm{e}^{-a|n|} - 1\right], \end{aligned} \tag{71}$$

where

$$a = \frac{\pi}{L}\Lambda. \tag{72}$$

The last sums in (71) are due the fact that, because of (68), the planes $n_1 = 0$, $n_2 = 0$, and $n_3 = 0$ have to be excluded from the sum over $\mathbb{Z}^3$ because they lead to eigenfunctions which are zero. For the same reason the axes $n_1 = n_2 = 0$, $n_1 = n_3 = 0$, $n_2 = n_3 = 0$ and the origin be excluded. Exclusion of the three planes (the term

$-3\sum_{\vec{n}\in\mathbb{Z}^2} \mathrm{e}^{-a|\vec{n}|}$ in (71)) corresponds to the exclusion of each axis twice instead of only once. The third term compensates for this, while the last one excludes the origin.

A method of calculation of the lattice sums in (71) is through the Poisson summation formula

$$\sum_{\vec{n}\in\mathbb{Z}^3} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^3} C_{\vec{m}}, \tag{73}$$

where $C_{\vec{m}}$ are the Fourier coefficients of f:

$$C_{\vec{m}} = \int \mathrm{d}^3x \mathrm{e}^{-2\pi\mathrm{i}\vec{m}\cdot\vec{x}} f(\vec{x}). \tag{74}$$

See also [19]. Applying (73) to (71), we find

$$\begin{aligned} E_{\mathrm{vac}}(\Lambda) = &-\frac{3}{2\pi^2}L^3\Lambda^{-4} + \frac{3}{2\pi^2}L^3\Lambda^{-4} \\ &- \frac{3}{4\pi}L^2\Lambda^{-3} + \frac{3}{8\pi}L\Lambda^{-2} \\ &- \frac{\pi^2}{2L}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2\right]^{-2} \\ &+ \frac{2\pi^4}{L}\left(\frac{\Lambda}{L}\right)^2\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2\right]^{-3} \\ &+ \frac{3\pi^2}{8L}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2\right]^{-3/2} \\ &- \frac{9\pi^4}{8L}\left(\frac{\Lambda}{L}\right)^2\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2\right]^{-5/2} \\ &- \frac{3\pi}{8L}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2m^2\right]^{-1} \\ &+ \frac{3\pi^3}{4L}\left(\frac{\Lambda}{L}\right)^2\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2m^2\right]^{-2}. \end{aligned} \tag{75}$$

We now expand the sums $\sum_{\vec{m}\neq\vec{0}}$ in (75) in the following way:

$$\begin{aligned} &\left[\left(\frac{\pi\Lambda}{L}\right)^2 + 4\pi^2|\vec{m}|^2\right]^{-s} \\ &= \left(4\pi^2|\vec{m}|^2\right)^{-s}\left(1 - \frac{s\Lambda^2}{4L^2|\vec{m}|^2} + \cdots\right). \end{aligned} \tag{76}$$

The unit term in (76) yields a contribution of type a_4L^{-1} in (3); the remaining terms provide the rest of the asymptotic series in (3) consisting of positive powers of Λ. We thus find

$$a_4 = -\frac{1}{32\pi^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}|\vec{m}|^{-4} + \frac{3}{64\pi}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}|\vec{m}|^{-3} - \frac{3}{32\pi}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}m^{-2}. \tag{77}$$

The last sum above is nothing but $2\zeta(2)$, where ζ stands for the Riemann zeta function, and the second one may be rewritten as the product of two independent sums by means of $\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}|\vec{m}|^{-s} = 4\zeta(s/2)\,\beta\left(\frac{s}{2}\right)$ (see, e.g., [20, 23]), where $\beta(s) = \sum_{j=0}^{\infty}(-1)^j/(2j+1)^s$. Then, using the result of Lukosz [24] for the first sum in (77) we obtain

$$a_4 = -0.0157322\ldots, \tag{78}$$

which is in accordance with the result obtained numerically in [20] (in fact we have obtained a_4 to a higher accuracy than shown). In addition, from (23b), the inner pressure is

$$p_{\mathrm{C}}^{\mathrm{inner}}(L) = \frac{a_4}{3L^4}. \tag{79}$$

By (78) and (79) we see that the force due to the interior is attractive. The repulsive character of the sphere (Sect. 3) suggests, however, that the same is true for the cube. This fact alone shows that this sign, if true, *must* be entirely due to the exterior force, a subtle problem to which we now turn.

5 The external problem for the cube

As remarked above, it is of great interest to consider also the outer problem for the cube. We will consider the cube K_L of side L concentric with a cube K_M, of side M, from which K_L is a subset ($M > L$ and M eventually goes to infinity at the end of calculation) and impose Dirichlet b.c. on K_L as well as K_M (see Sect. 1). Unfortunately, the solution of the external Casimir problem for the cube with Dirichlet b.c. cannot be constructed out of the functions of the form (68), because the continuity conditions on several planes cannot be satisfied simultaneously. However, the form of solutions (68), which are naturally adapted to the internal geometry of the cube, suggest splitting the region $K_M\backslash K_L$ into 26 subregions bounded by the planes containing the faces of the cube. We may require the $u_n(\vec{x})$ to vanish on the boundaries of these subregions, including the original requirement of vanishing on the faces of the internal and external cubes. If we do so, the resulting problem is explicitly solvable in terms of the set (68). Of course, this Ansatz introduces additional stresses in the region $K_M\backslash K_L$. We shall comment on these restrictions at the end of this section.

Then we have that the 26 subregions which compose $K_M\backslash K_L$ are of three topologically distinct kinds (with both cubes centered in the origin):

(1) a rectangular box with two sides L and one $(M-L)/(2)$ – with multiplicity 6;
(2) a rectangular box with two sides $(M-L)/(2)$ and one L (with multiplicity 12) – the contribution of the edges;
(3) a cube of sides $(M-L)/(2)$ (with multiplicity 8) – the contribution of the corners.

The Casimir energy of each of these regions can be obtained along the same lines of the calculation above outlined for the inner cube (see [19]). Then we obtain that the regions of type (3) do not contribute, i.e., there are no contributions of the corners, either to the energy or the pressure, in the limit $M \to \infty$. The total contribution of the regions of type (1) is

$$\begin{aligned}
&E_1(L,M) \\
&= -\frac{3L^2(M-L)}{32\pi^2} \sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2L^2 + m_3^2\frac{(M-L)^2}{4}\right]^2} \\
&+ \frac{3L(M-L)}{32\pi} \sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4}\right]^{3/2}} \\
&+ \frac{3}{32\pi L} \sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}} \frac{1}{[m_1^2+m_2^2]^{3/2}} - \frac{\pi}{16}\left(\frac{2}{L} + \frac{2}{M-L}\right), \qquad (80)
\end{aligned}$$

and for the regions of type (2) the total contribution is

$$\begin{aligned}
&E_2(L,M) = -\frac{3L(M-L)^2}{32\pi^2} \qquad (81) \\
&\times \sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4} + m_3^2\frac{(M-L)^2}{4}\right]^2} \\
&+ \frac{3L(M-L)}{16\pi} \sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4}\right]^{3/2}} \\
&+ \frac{3}{8\pi(M-L)} \sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}} \frac{1}{[m_1^2+m_2^2]^{3/2}} - \frac{\pi}{8}\left(\frac{1}{L} + \frac{4}{M-L}\right).
\end{aligned}$$

Then, by (23c) and taking into account that $\mathrm{d}V_{\text{outer}} = 6\mathrm{d}V_1 + 12\mathrm{d}V_2 + 8\mathrm{d}V_3$, where V_i is the volume of type (i) region, and that in the thermodynamic limit $\mathrm{d}V_3 \to 0$ ($\mathrm{d}V_3/V_3 \propto 1/M$), we obtain

$$\begin{aligned}
&p_{\mathrm{C}}^{\text{outer}}(L) \qquad (82) \\
&= -\lim_{M\to\infty} \frac{1}{3M(M-2L)} \frac{\partial}{\partial L} \left[E_1(L,M) + E_2(L,M)\right].
\end{aligned}$$

It may also be verified explicitly that regions of type (3) do not contribute to the energy, or to the pressure, in the thermodynamic limit. From (82) we see that type (1) regions do not contribute to the outer pressure, while the edges contribution is given only by the first term in (81):

$$\begin{aligned}
&p_{\mathrm{C}}^{\text{outer}}(L) = \lim_{M\to\infty} \left\{ \frac{\partial}{\partial L} \left(\frac{L}{32\pi^2} \right.\right. \qquad (83) \\
&\left.\left. \times \sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_3\neq 0}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4} + m_3^2\frac{(M-L)^2}{4}\right]^2} \right) \right\}.
\end{aligned}$$

It follows from (83) that (see later):

$$p_{\mathrm{C}}^{\text{outer}}(L) = \frac{1}{32\pi^2} \left(\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} m^{-4} \right) \frac{\partial}{\partial L} L^{-3} = -\frac{\pi^2}{480} L^{-4}. \qquad (84)$$

There is no contribution in the thermodynamic limit to (83) from the sum over $m_2 \neq 0$ or $m_3 \neq 0$, or both. Indeed, the term $m_1 = 0$ in (83) does not contribute as $M \to \infty$, as one sees easily, and

$$\begin{aligned}
&\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_1\neq 0\\ m_2\neq 0, m_3\neq 0}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4} + m_3^2\frac{(M-L)^2}{4}\right]^2} \\
&\leq \left(\frac{2}{L^7(M-L)}\right)^{1/2} \sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ m_1\neq 0\\ m_2\neq 0, m_3\neq 0}} \frac{1}{[m_1^2+m_2^2+m_3^2]^{7/4}}. \qquad (85)
\end{aligned}$$

If only $m_2 \neq 0$ (or $m_3 \neq 0$)

$$\begin{aligned}
&\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ m_1\neq 0, m_2\neq 0}} \frac{1}{\left[m_1^2L^2 + m_2^2\frac{(M-L)^2}{4}\right]^2} \\
&\leq \left(\frac{2}{L^3(M-L)}\right) \sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ m_1\neq 0, m_2\neq 0}} \frac{1}{[m_1^2+m_2^2]^{3/2}}, \qquad (86)
\end{aligned}$$

for $L \leq 1$, $M > L$. It may be checked that these bounds suffice to show that the contributions of the above sums to the limit in (83) is zero. The only surviving term in the sum in (83) is thus $m_2 = m_3 = 0$, which leads to (84).

It is most important to note that the edges' contribution to the Casimir pressure is greater than the inner pressure in absolute value: by (79), (84) and (23a)

$$p_{\mathrm{C}}(L) = \left(-0.005244 + \frac{\pi^2}{480}\right) L^{-4} = 0.015317 L^{-4}. \qquad (87)$$

The Casimir pressure is thus repulsive, and the net result is that edge effects *determine the sign of the force*. We shall return to this point in the conclusion.

We now comment on our Ansatz. We have replaced the original Hilbert space of L^2-functions with Dirichlet b.c. in the inner and outer boundaries with a direct sum of 26 spaces, upon introduction of additional Dirichlet b.c. on planes which are extensions of the cube's faces to the region $K_M \backslash K_L$. In this region all extra stresses are *parallel* to the cube's faces, and for this reason the Casimir pressure is insensitive to their inclusion. We have verified this assertion in Appendix A by introducing an extra Dirichlet plane orthogonal to a system of parallel plates in the inner region. The proof generalizes to an arbitrary finite number of such planes provided they are placed symmetrically to some plane orthogonal to the z-axis, thus not introducing an extraneous length in the original problem. Since the parallel plates are a soluble "limiting case" of the cube, we (strongly) believe that our Ansatz provides the exact solution for the cube.

6 Conclusion and open problems

In this paper we have introduced a mathematically precise framework for the Casimir pressure, by associating it to the cutoff-independent part of the Ramanujan sum of the (divergent) series for the Casimir energy. Our ideas have precursors in [10,8]. We illustrated the framework by parallel plates, the sphere and the interior problem of the cube.

In Sect. 5 we introduced an Ansatz to calculate the exterior Casimir pressure for the cube. We discussed why we (strongly) believe that it is the exact solution for the cube. If our conjecture is right, the calculation of Sect. 5 also provides an explanation for the sign of the force: it is due to a competition between the inner and outer pressures, in which the latter is positive and larger than the former in absolute value, because, as remarked in Sect. 5, the thermodynamic limit selects a set of modes different from the inner ones[2], with a large positive contribution from the edges. The edges reflect the passage from the infinitely extended parallel plates to a compact region, i.e., by folding. If this folding were smooth, i.e., for any smooth approximation to the cube, it would be accompanied by nonzero *curvature*. At the other extreme – *uniform* nonzero curvature – we have the sphere. Here, however, curvature effects appear less directly, reflecting themselves in the appearance of the Neumann functions in the external problem. It is an interesting open problem to understand more clearly the role of curvature (of various kinds, e.g. Riemannian, the mean and Gaussian curvatures) in the Casimir effect for general compact manifolds with boundary (see also [10]).

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Appendix A

In this appendix we consider the problem of the parallel plates with an orthogonal (Dirichlet) plane introduced in the inner region and prove that the Casimir *pressure* on the plates is the same that for the problem without the Dirichlet plane (see Sect. 2).

Consider the two parallel plates placed at $z = 0$ and $z = d$, and the inner Dirichlet plane at $y = 0$, $0 \le z \le d$. The outer problem is the same as the one we worked out in Sect. 2, and it does not contribute. The inner Casimir problem is now split into two regions: $K_1^{\text{inner}} = \{\vec{x} \in \mathbb{R}\times[0,\infty)\times[0,d]\}$ and $K_2^{\text{inner}} = \{\vec{x} \in \mathbb{R}\times(-\infty,0]\times[0,d]\}$. In order to calculate the Casimir energy of these regions it is necessary to consider a finite region in the x–y plane with area $A = L_1L_2$ (the whole area of the plates), and take the limit $L_1, L_2 \to \infty$ at the end. Then, both K_1^{inner} and K_2^{inner} are given by boxes of sides L_1 (along the x-axis), $L_2/2$ (along the y-axis) and d, so that we can proceed in the same way as in Sect. 4, obtaining for the Λ-independent term of the series (3) for the region K_1^{inner}:

$$\begin{aligned}
&E_1^{\text{inner}}(L_1,L_2,d) \qquad\qquad (A.1)\\
&= -\frac{L_1L_2d}{64\pi^2}\sum_{\substack{\vec{m}\in\mathbb{Z}^3\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2L_1^2+m_2^2\frac{L_2^2}{4}+m_3^2d^2\right]^2}\\
&+\frac{L_1L_2}{128\pi}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2L_1^2+m_2^2\frac{L_2^2}{4}\right]^{3/2}}\\
&+\frac{L_1d}{64\pi}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2L_1^2+m_2^2d^2\right]^{3/2}}\\
&+\frac{L_2d}{128\pi}\sum_{\substack{\vec{m}\in\mathbb{Z}^2\\ \vec{m}\neq\vec{0}}}\frac{1}{\left[m_1^2\frac{L_2^2}{4}+m_2^2d^2\right]^{3/2}}-\frac{\pi}{96}\left(\frac{1}{L_1}+\frac{2}{L_2}+\frac{1}{d}\right),
\end{aligned}$$

and, obviously, $E_2^{\text{inner}}(L_1,L_2,d)$ has the same form.

Now we can calculate the inner pressure by means of (23b), where V_{inner} must be taken as the whole interior volume: $V_{\text{inner}} = L_1L_2d$. Thus, we have

$$\begin{aligned}
&p_{\mathrm{C}}(d) = p_{\mathrm{C}}^{\text{inner}}(d)\\
&= \lim_{L_1,L_2\to\infty}\left(-\frac{\partial E_1^{\text{inner}}(L_1,L_2,d)}{\partial V_{\text{inner}}}-\frac{\partial E_2^{\text{inner}}(L_1,L_2,d)}{\partial V_{\text{inner}}}\right)\\
&= \lim_{L_1,L_2\to\infty}\left(-\frac{2}{L_1L_2}\frac{\partial}{\partial d}E_1^{\text{inner}}(L_1,L_2,d)\right), \qquad (A.2)
\end{aligned}$$

where we have taken into account that $E_1^{\text{inner}} = E_2^{\text{inner}}$.

From (A.1) and (A.2) we see that the only term which contributes to the pressure at the thermodynamic limit is the first term at the right side of (A.1); in fact, only the term $m_1 = m_2 = 0$ contributes (that the terms with $m_1 \neq 0$ or $m_2 \neq 0$, or both, do not contribute can be proved just in the same way as in Sect. 5 – see (85) and (86)). Then we obtain

$$p_{\mathrm{C}}(d) = \frac{1}{16\pi^2}\zeta(4)\frac{\partial}{\partial d}d^{-3} = -\frac{\pi^2}{480}d^{-4}, \qquad (A.3)$$

which is the same result obtained in the case without the inner Dirichlet plane!

The above result shows that the introduction of additional stresses parallel to the physical plates do not modify the pressure on these plates. Finally, it is easy to see that the above proof generalizes in a trivial way to the case in which the Dirichlet plane is at the outer region, and also to the case in which we have a finite number of Dirichlet planes in the inner region provided these planes are disposed symmetrically with respect to some plane orthogonal to the z-axis.

[2] See also [33] for a discussion (different of ours) of the different roles of the inner and outer modes of the Casimir problem

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